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A lower bound for the ground-state energy of many particles moving in one dimension

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Abstract. We use a technique due to Spruch to investigate lower bounds for the eigenvalues of the following problems: (1) one particle in one dimension; (2) one particle moving in two and three dimensions; (3) N particles moving in one dimension. In all cases existence theorems for bound states are obtained.

1. Introduction

A very simple lower bound for the energy E of the ground state of a Schrödinger particle moving in one dimension under the influence of a potential V was derived by Spruch (1961). It states that $E > E_\delta$ where E_δ is the energy of the bound state of the particle in the presence of a (δ-function) potential $V_\delta(x) = -I\delta(x)$ with $I = \int |V_-|\,dx$ (the integral of the negative part V_- of the potential V).

Now, the N-body problem for a system of identical particles of mass m moving in one dimension with a two-body interaction $\lambda \sum_{i<j=1}^{N} \delta(x_i - x_j)$ is also solvable (McGuire 1964). In particular for $\lambda < 0$ the energy of the (unique) bound state is explicitly given by $E_\delta^{(N)} = -\lambda^2(m/24\hbar^2)N(N^2-1)$. It is therefore natural to conjecture the validity of the analogue bound

$$E^{(N)} > E_\delta^{(N)} \tag{1}$$

for the energy $E^{(N)}$ of the ground state of a system of N identical particles in one dimension interacting via a two-body potential $V(x_i - x_j)$. Here $\lambda = I = \int |V_-(x)|\,dx$. For $N > 2$, however, this result is far less trivial, as the original variational argument of Spruch (1961) is not sufficient. In this paper we show that (1) is valid for a large class of two-body interactions. The proof we present here uses the elegant log-concavity techniques in Lieb and Simon (1978).

For dimension $\nu = 2$ or 3 we can establish only a much weaker result based on an extension of Spruch's (1961) result to this case combined with the Hall–Post theorem (Hall and Post 1967).

This paper is organised as follows. In § 2 we revisit and extend for higher dimension the results of Spruch (1961). As a byproduct of our method we rederive some results of Simon (1976) and Klaus (1977) concerning necessary conditions for the existence of bound states of a particle moving in a potential in $\nu = 1$ dimension. We obtain lower bounds for the energy of the ground state and, in the case of even potentials, for the first excited state of the particle also. For $\nu = 3$ the Jost–Pais conditions (Jost and Pais 1951) are rederived. For dimensions $\nu = 2, 3$ lower bounds are obtained for the energy of the ground state of a particle in a potential. The proof of (1) appears in § 3. In this section we also extend for $N>2$ the results of Simon (1976) and Klaus (1977) concerning the absence of bound states in $\nu=1$ for an arbitrarily weak and globally repulsive ($\int V\,\mathrm{d}x>0$) potential. We also discuss some lower bounds for N particle systems.

2. One particle in a potential

Assuming that a particle moving in a potential has a bound state we shall obtain a lower bound for its energy. In $\nu=1$ dimension this is done by using a technique due to Spruch (1961). In $\nu=2$, 3 dimensions we first have to extend Spruch's technique and then use it to obtain lower bounds. In order to avoid technicalities that could obscure the method, unless otherwise stated we shall assume $V\in C_0^\infty(R^\nu)$ (i.e. infinitely differentiable and of compact support), even though the results are valid under far less stringent assumptions. The mathematically inclined reader will find no difficulty in supplying more general conditions.

We start with the one-dimensional case for which we are able to provide lower bounds also for the first excited state if the potential is parity invariant. In one dimension we use the following theorem.

Theorem 1. (Spruch 1961). Let $V_-(x)$ be the negative part of $V(x)$. Then, of all potentials $V(x)$ with the same value of $I=\int_{-\infty}^{\infty}|V_-(x)|\,\mathrm{d}x$, the δ-function potential has the lowest energy. Hence, a lower bound for the ground-state energy is $\varepsilon^{(0)}=-(\mu/2\hbar^2)I^2$ (μ is the mass of the particle or the reduced mass in the two-body case). Since the basic argument used in the proof will be repeatedly used in this work we shall present it here.

Proof. Let ψ be the exact normalised ground-state wavefunction of H_0+V. So

$$E^{(0)}=(\psi,[H_0+V]\psi)=(\psi,H_0\psi)+\int_{-\infty}^{\infty}|\psi(x)|^2V(x)\,\mathrm{d}x. \tag{2}$$

Let $|\psi(a)|$ be the maximum value of $|\psi(x)|$. Then

$$E^{(0)}\geqslant(\psi,H_0\psi)-I|\psi(a)|^2=(\psi,H_0\psi)+(\psi,V_\delta\psi) \tag{3}$$

where

$$V_\delta=-I\delta(x-a). \tag{4}$$

And, from the variational principle, $(\psi,(H_0+V_\delta)\psi)\geqslant\varepsilon^{(0)}$, thus proving the theorem.

Now, suppose the potential is parity invariant, $V(x)=V(-x)$, the above technique provides a necessary condition for the existence of a first excited state and a lower bound for its energy. This is the content of the following theorems 2 and 3.

Theorem 2. Given a particle moving in a one-dimensional parity-invariant potential $V(x)$, a necessary condition for the existence of a (bound) *first* excited state is

$$\frac{\mu}{\hbar^2}\int_{-\infty}^{\infty}|x|V_-(x)\,\mathrm{d}x<-1. \tag{5}$$

Proof. Assume that there exists a first excited state of H_0+V. Let $\psi^{(-)}$ be its normalised wavefunction (the $^{(-)}$ indicates negative parity). Then

$$E^{(1)}=(\psi^{(-)},H\psi^{(-)})=(\psi^{(-)},H_0\psi^{(-)})+\int_{-\infty}^{\infty}\mathrm{d}x|\psi^{(-)}(x)|^2V(x)\,\mathrm{d}x. \tag{6}$$

Let $x=a$ be the point where $|x^{-1}|\psi^{(-)}(x)|^2|$ is maximum. By symmetry we have $|a^{-1}|\psi^{(-)}(a)|^2|=|-a^{-1}|\psi^{(-)}(-a)|^2|$ and so

$$\begin{aligned}E^{(1)}&\geqslant(\psi^{(-)},H_0\psi^{(-)})+\frac{1}{2}\left(\frac{|\psi^{(-)}(a)|^2}{|a|}+\frac{|\psi^{(-)}(-a)|^2}{|-a|}\right)\int_{-\infty}^{\infty}|x|V_-(x)\,\mathrm{d}x\\&=(\psi^{(-)},H_0\psi^{(-)})+(\psi^{(-)},V_{2,\delta}\psi^{(-)})\end{aligned} \tag{7}$$

where

$$V_{2,\delta}=\left(\frac{1}{2|a|}\int_{-\infty}^{\infty}|x|V_-(x)\,\mathrm{d}x\right)[\delta(x+a)+\delta(x-a)]. \tag{8}$$

From the variational principle

$$(\psi^{(-)},H_0\psi^{(-)})+(\psi^{(-)},V_{2,\delta}\psi^{(-)})\geqslant E^{(1)}_{2,\delta} \tag{9}$$

($E^{(1)}_{2,\delta}$ is the energy of the *first* excited state of $H_0+V_{2,\delta}$) so that

$$E^{(1)}\geqslant E^{(1)}_{2,\delta}. \tag{10}$$

A simple explicit computation shows that the Hamiltonian $H_0+V_{2,\delta}$ will have a bound first excited state only if condition (5) is satisfied, and so the theorem is proved.

Now, a lower bound for the energy of the first excited state of a particle moving in a globally attractive (i.e. $\int V(x)\,\mathrm{d}x<0$) symmetric potential is obtained by using the fact that when $a\to\infty$ there is a degeneracy of the ground state with the first excited state of the Hamiltonian

$$H=p^2/2\mu+B[\delta(x+a)+\delta(x-a)].$$

Actually the ground-state energy $E^{(0)}_{2,\delta}(a)$ increases monotonically and the first excited state energy $E^{(1)}_{2,\delta}(a)$ decreases monotonically with increasing a, degenerating when $a\to\infty$ to

$$E^{(0)}_{2,\delta}(\infty)=E^{(1)}_{2,\delta}(\infty)=-(\mu/2\hbar^2)B^2. \tag{11}$$

Thus we have theorem 3.

Theorem 3. A lower bound for the energy of the first excited state of a particle moving in a one-dimensional *symmetric* potential $V(x)$ is

$$\varepsilon^{(1)}=-\tfrac{1}{8}(\mu/\hbar^2)I^2. \tag{12}$$

Proof. Let $\psi^{(-)}$ be the normalised wavefunction of the first excited state of $H=H_0+V$. Then its energy is

$$E^{(1)}=(\psi^{(-)},H\psi^{(-)})=(\psi^{(-)},H_0\psi^{(-)})+\int_{-\infty}^{\infty}|\psi^{(-)}(x)|^2V(x)\,\mathrm{d}x.$$

Let $x=\pm a$ be the points of maximum of $|\psi^{(-)}(x)|$. Hence, as $\psi^{(-)}(a)=-\psi^{(-)}(-a)$,

$$E^{(1)} \geqslant (\psi^{(-)}, H_0\psi^{(-)}) + \tfrac{1}{2}[|\psi^{(-)}(a)|^2 + |\psi^{(-)}(-a)|^2] \int_{-\infty}^{\infty} V_-(x)\,dx$$

$$= (\psi^{(-)}, H_0\psi^{(-)}) + (\psi^{(-)}, V'_{2,\delta}\psi^{(-)})$$

where

$$V'_{2,\delta} = -\tfrac{1}{2}I[\delta(x+a)+\delta(x-a)].$$

Therefore the *first excited* state of $H_0 + V'_{2,\delta}$ provides a lower bound for the first excited state of H. And using equation (11), equation (12) follows, thus proving the theorem.

Remark. The lower bound (12) to the energy of the first excited state of a symmetric potential in one dimension coincides with the lower bound (16) to the energy of the ground state of a spherical potential in $\nu = 2, 3$ dimensions.

Now, if we take into account the contribution of the repulsive part of the potential, a better lower bound can be obtained using the central idea of replacing the potential by conveniently chosen δ functions. To illustrate this technique we present below a variational proof of the following theorem, first proved by Simon (1976) and Klaus (1977).

Theorem 4. Let $V_+(x) = V(x) - V_-(x)$, then if $\int_{-\infty}^{\infty} V(x)\,dx > 0$ and supp V_- is compact, $H = H_0 + \lambda V(x)$ has no bound states for $\lambda > 0$ sufficiently small.

Proof. Let us first consider the case where supp V_+ is also compact. Suppose $H_0 + \lambda V$ has a ground state of negative energy E for all $\lambda > 0$. Let $\psi_\lambda(x)$ be its normalised wavefunction. Now, $\psi_\lambda(x)$ is a positive and continuous (in fact twice differentiable) function. This implies that there exists $x_+(\lambda) \in \text{supp } V_+$ such that $\psi_\lambda(x_+(\lambda)) = \min_{x\in \text{supp } V_+} \psi_\lambda(x)$. Let now $x_-(\lambda)$ be such that $\psi_\lambda(x_-(\lambda)) = \max_{x\in\text{supp } V_-} \psi_\lambda(x)$, then

$$\begin{aligned} E &= (\psi_\lambda, (H_0+\lambda V)\psi_\lambda) \\ &= (\psi_\lambda, H_0\psi_\lambda) + \lambda \int_{-\infty}^{\infty} V(x)|\psi_\lambda(x)|^2\,dx \\ &> (\psi_\lambda, H_0\psi_\lambda) - \lambda I|\psi_\lambda(x_-(\lambda))|^2 + \lambda I'|\psi_\lambda(x_+(\lambda))|^2 \\ &= (\psi_\lambda, H_0\psi_\lambda) + (\psi_\lambda, \lambda V_{\delta,\lambda}\psi_\lambda) \end{aligned} \tag{13}$$

where

$$V_{\delta,\lambda}(x) = -I\delta(x - x_-(\lambda)) + I'\delta(x - x_+(\lambda))$$

$$I' = \int_{-\infty}^{\infty} V_+\,dx > I = \int_{-\infty}^{\infty} |V_-|\,dx.$$

Now, a simple calculation shows that when $\lambda \to 0$ the Hamiltonian $H_0 + \lambda V_{\delta,\lambda}$ has no bound states provided $x_+(\lambda)$ and $x_-(\lambda)$ vary over fixed compact sets.

If supp V_+ is not compact, let $V_+^R = V^R - V_-$ where $V^R(x) = V(x)$ if $|x| \leqslant R$ and $V^R(x) = 0$ for $|x| > R$. Now, V_+^R has compact support and, for sufficiently large R, $\int V^R(x)\,dx > 0$ and so $H_0 + \lambda V^R$ has no bound state for $\lambda > 0$ sufficiently small. Since $V \geqslant V^R$, the same conclusion applies to $H_0 + \lambda V$, thus concluding the proof.

Remark. If the supp V_- is not compact, then some restriction on its tail has to be imposed for if $V(x) \sim -1/x^{1+\varepsilon}(0<\varepsilon<1)$ as $x \to \infty$, then $H_0 + \lambda V(x)$ has an infinite number of bound states for any $\lambda > 0$ (Simon 1970, Perez *et al* 1986).

We shall now extend Spruch's technique to obtain necessary conditions for the existence of bound states for a particle moving in two and three dimensions. This is the content of theorem 5†.

Theorem 5. A particle moving in a central potential $V(r)$ in $\nu = 2, 3$ dimensions will have a bound state with an angular momentum l only if

$$\frac{2\mu}{\hbar^2} \int_0^\infty |V_-(r)| r \, \mathrm{d}r \geq [2l + (\nu - 2)]. \tag{14}$$

Proof. Let ψ_l be the normalised wavefunction of the lowest bound state of $H_0 + V(r)$ with angular momentum l. Let E_l be its energy. Then

$$E_l = (\psi_l, H\psi_l) = (\psi_l, H_0\psi_l) + \int \mathrm{d}\Omega \int_0^\infty |\psi_l(r)|^2 V(r) r^{\nu-1} \, \mathrm{d}r.$$

Let $r = a$ be the point where $r^{\nu-2}|\psi_l(r)|^2$ is maximum. Then

$$E_l \geq (\psi_l, H_0\psi_l) + (\nu - 1) 2\pi a^{\nu-2} |\psi_l(a)|^2 \int_0^\infty V_-(r) r \, \mathrm{d}r$$

$$= (\psi_l, H_0\psi_l) + (\psi_l, V_\delta \psi_l)$$

where

$$V_\delta = -\frac{\lambda}{a} \delta(r - a)$$

with

$$\lambda = \int_0^\infty |V_-(r)| r \, \mathrm{d}r.$$

Now, if E_l^δ is the energy of the lowest bound state (with angular momentum l) of $H_0 + V_\delta$, from the variational principle,

$$E_l \geq E_l^\delta. \tag{15}$$

An elementary calculation shows that $H_0 + V_\delta$ has a bound state of angular momentum l only if condition (14) is satisfied and this completes the proof.

The approach used supplies only the lower bound zero on the ground state of potentials which satisfy equation (14). Thus, the bounding potential is $V_\delta = (-\lambda/a)\delta(r - a)$, for some unknown value of a. Since V_δ generates an energy of $-\infty$ for $a = 0$ and an energy zero for $a = \infty$, and since we do not know the appropriate value of a, we can say nothing about the binding energy. We will return to the question of a lower bound in theorem 7.

From theorem 5 we see that for $\nu = 3$ a necessary condition for the existence of a bound state of a particle moving in a central potential $V(r)$ is that $(2\mu/\hbar^2) \int_0^\infty r|V_-(r)| \, \mathrm{d}r > 1$ which is the necessary condition of Jost and Pais (1951).

† Extension to dimension $\nu > 3$ is straightforward.

Remark. For $\nu=2$ theorem 5 suggests that any two-dimensional well could have at least one bound state. In fact, a sufficient condition for the existence of at least one bound state is that the well be globally attractive: $\int V(r)\,\mathrm{d}^2r<0$ (Simon (1976); see Coutinho *et al* (1983) for a variational proof that extends to N particles). If the two-dimensional potential is not globally attractive we have the following theorem (Simon 1976, Klaus 1977).

Theorem 6. If $\int V(r)\,\mathrm{d}^2r>0$ and supp V_- is compact then the two-dimensional Hamiltonian $H=H_0+\lambda V(r)$ has no bound states for λ sufficiently small. The proof follows closely that of the corresponding one-dimensional case (theorem 4) so we shall not repeat it here.

Remark. If supp V_- is not compact some restrictions must again be imposed on the decay rate of V_- since if $V(r)\leqslant -\frac{1}{4}(r\ln r)^{-2}$ for large r, there are infinitely many bound states (Perez *et al* 1986).

The following theorem extends theorem 3 for $\nu=2,3$ dimensions.

Theorem 7. A lower bound for the energy of the point spectrum of a particle moving in a spherical potential $V(r)$ in $\nu=2,3$ dimensions is

$$\tilde{\varepsilon}=-(\mu/2\hbar^2)\tilde{\lambda}^2 \tag{16}$$

where

$$\tilde{\lambda}=\int_0^\infty |V_-(r)|\,\mathrm{d}r. \tag{17}$$

Proof. Let ψ be an eigenstate of $H=H_0+V(r)$ and E its energy. Then

$$E=(\psi,H\psi)=(\psi,H_0\psi)+\int \mathrm{d}\Omega\int r^{\nu-1}\,\mathrm{d}r\,V(r)|\psi(r)|^2.$$

Let $r=a$ be the point of maximum of $r^{\nu-1}|\psi(r)|^2$. Then

$$E\geqslant(\psi,H_0\psi)+(\nu-1)2\pi a^{\nu-1}|\psi(a)|^2\tilde{\lambda}$$
$$=(\psi,H_0\psi)+(\psi,\tilde{V}_\delta\psi)$$

where

$$\tilde{V}_\delta=-\tilde{\lambda}\delta(r-a)$$

and, from the variational principle,

$$E\geqslant\tilde{E}_\delta$$

where $\tilde{E}_\delta$ is the eigenvalue of $H_0+\tilde{V}_\delta$.

Now, when $a\to\infty$ all the eigenvalues of $H_0+\tilde{V}_\delta$ degenerate monotonically at the minimum value $\tilde{\varepsilon}=-(\mu/2\hbar^2)\tilde{\lambda}^2$ and this completes the proof.

Note that, as opposed to $V_\delta = -(\lambda/a)\delta(r-a)$, the potential $\tilde{V}_\delta = -\tilde{\lambda}\delta(r-a)$ gives a finite binding energy for all a, and can therefore supply a lower bound $\tilde{\varepsilon}$ on the energy. On the other hand, since the energy associated with $\tilde{V}_\delta$ ranges from its value at $a=0$, $\tilde{\varepsilon}(a=0)$, which is negative, to $\tilde{\varepsilon}(a=\infty) = -(\mu/2\hbar^2)\tilde{\lambda}^2$, this approach cannot generate the necessary condition for the existence of a bound state given by equation (14). (In fact, it generates the necessary condition for the existence of a bound state with an energy below $\varepsilon(a=0)$.)

3. The N-particle case

We now turn to the N-body problem. Sufficient conditions for the existence of an N-particle bound state were given by Coutinho *et al* (1983, 1984) and by Perez *et al* (1985) for $\nu = 1, 2$ and 3. We shall extend Spruch's technique to obtain lower bounds for the energy of an N-particle bound state. This can be done only for the case of N particles moving in one dimension because it requires exactly solving the problem of N particles interacting via a δ potential.

In one dimension if the interparticle potential is globally attractive there always exists a bound state of N particles. The lower bound for its energy is given by the following theorem.

Theorem 8. A lower bound for the ground-state energy of N identical bosons of mass m moving in one dimension and interacting via a globally attractive potential is given by the ground-state energy of N particles interacting via the potential $V_\delta = -I\delta(x_i - x_j)$ ($I = \int_{-\infty}^{\infty} |V_-(x)|\, dx$). Hence (McGuire 1964) a lower bound is

$$\varepsilon = -(m/24\hbar^2)N(N^2-1)I^2. \tag{18}$$

Proof. Let Ψ be the exact symmetric normalised wavefunction of the ground state of $H = H_0 + \Sigma_{i>j}^{N} V(x_i - x_j)$. Then its energy is

$$E = (\Psi, H_0\Psi) + \frac{N(N-1)}{2}\int d\rho\, \varphi(\rho) V(\rho) \tag{19}$$

where $\varphi(\rho) = \int |\Psi(x,\rho)|^2\, dx$, $\rho = x_1 - x_2$ and x stands for all the other coordinates.

Let $\rho = \rho_0$ be the point of maximum of $\varphi(\rho)$. Then,

$$\begin{aligned} E &\geqslant (\Psi, H_0\Psi) - \frac{N(N-1)}{2}\varphi(\rho_0) I \\ &= (\Psi, H_0\Psi) - \frac{N(N-1)}{2}\int_{-\infty}^{\infty} d\rho\, \varphi(\rho)\frac{I}{2}(\delta(\rho-\rho_0) + \delta(\rho+\rho_0)) \\ &= (\Psi, (H_0 + V_{\delta,\rho_0})\Psi) \end{aligned} \tag{20}$$

where

$$V_{\delta,\rho_0} = -\tfrac{1}{2}\sum_{i<j} I\delta(|x_i - x_j| - \rho_0). \tag{21}$$

Now, the case $\rho_0 = 0$ is exactly solvable (McGuire 1964) and we now show that the minimum energy occurs at this value of ρ_0.

Theorem 9. The ground-state energy of the Hamiltonian

$$H=\sum\frac{P_i^2}{2m}-\frac{1}{2}\sum_{i<j} I\delta(|x_i-x_j|-\rho_0) \tag{22}$$

is a minimum when $\rho_0=0$.

Proof. The proof is adapted from an article by Lieb and Simon (1978) on the monotonicity of the electronic contribution to the Born-Oppenheimer energy.

Consider a Hamiltonian $H(\rho_0)=H_0+V_{\rho_0}$ depending on a parameter ρ_0. Then the ground-state energy is given by

$$E(\rho_0)=-\lim_{\beta\to\infty}\beta^{-1}\log(\psi,\exp(-\beta H(\rho_0))\psi) \tag{23}$$

where ψ is any function not orthogonal to the ground state. We shall take $\psi=\exp(-x^2)$ positive for any x value and hence not orthogonal to the ground state.

Using the Trotter formula we can write

$$(\psi,\exp(-\beta H(\rho_0))\psi)=\lim_{n\to\infty}\left\{\psi,\left[\exp\left(-\frac{\beta H_0}{n}\right)\exp\left(-\frac{\beta}{n}V_{\rho_0}\right)\right]^n\psi\right\}.$$

For n finite we have

$$\begin{aligned}&\left\{\psi,\left[\exp\left(-\frac{\beta}{n}H_0\right)\exp\left(-\frac{\beta}{n}V_{\rho_0}\right)\right]^n\psi\right\}\\&\quad=\left[\psi,\exp\left(-\frac{\beta}{n}H_0\right)\exp\left(-\frac{\beta}{n}V_{\rho_0}\right)\ldots\exp\left(-\frac{\beta}{n}H_0\right)\exp\left(-\frac{\beta}{n}V_{\rho_0}\right)\psi\right]\\&\quad=C\int \mathrm{d}x_1\ldots\mathrm{d}x_{n+1}\exp(-x_{n+1}^2)\\&\qquad\times\prod_{k=1}^{n-1}\left[\exp\left(-\frac{n(x_{k+1}-x_k)^2}{2\beta}\right)\exp\left(-\frac{\beta}{n}V_{\rho_0}(x_k)\right)\right]\exp(-x_1^2)\end{aligned} \tag{24}$$

C being a global normalisation constant.

Now, for the Hamiltonian (22) we have

$$V_{\rho_0}=-\frac{I}{4}\sum_{\substack{i,j\\(i\neq j)}}\delta(|x_i-x_j|-\rho_0)$$

so that

$$\exp\left(-\frac{\beta}{n}V_{\rho_0}(x_k)\right)=\exp\left(-\frac{\beta}{n}\frac{I}{4}\sum_{i\neq j}\delta(|x_i^k-x_j^k|-\rho_0)\right).$$

The δ functions may be approximated by bell-shaped functions U_α, and so

$$\exp\left(-\frac{\beta}{n}V_{\rho_0}(x_k)\right)=\lim_{\alpha\to\infty}\prod_{\substack{i,j\\(i\neq j)}}\exp\left(-\frac{\beta}{n}U_\alpha(|x_i^k-x_j^k|-\rho_0)\right).$$

Now, the exponential term on the right-hand side of this last equation is a symmetric decreasing function in $\xi_k^{ij}=|x_i^k-x_j^k|-\rho_0$ and therefore is an integral with positive weight $\mathrm{d}\mu_\alpha$ of characteristic functions of intervals (Simon 1979). That is, we can write

$$\exp\left(-\frac{\beta}{n}V_{\rho_0}(x_k)\right)=\lim_{\alpha\to\infty}\prod_{\substack{i,j\\i\neq j}}\int \mathrm{d}\mu_\alpha(\xi_k^{ij})\chi_{ij}^k$$

where χ_{ij}^k is a characteristic function of an interval and therefore log concave ($\xi_k^{ij} = |x_i^k - x_j^k| - \rho_0$). Substitution of the above expression in (24), after integration on the variables $x_1, \ldots, x_{n+1}$, results in

$$\left\{\psi, \left[\exp\left(-\frac{\beta}{n} H_0\right) \exp\left(-\frac{\beta}{n} V_{\rho_0}\right)\right]^n \psi\right\} = \lim_{\alpha \to \infty} \int \prod_{i \neq j}^{N} \prod_{k=1}^{n} \mathrm{d}\mu(\xi_k^{ij}) f(\boldsymbol{\xi}, \rho_0) \tag{25}$$

where $\boldsymbol{\xi}$ denotes all the ξ_k^{ij}.

Now, by the Prékopa theorem (Prékopa 1971), $f(\boldsymbol{\xi}, \rho_0)$ is log concave in ρ_0 and so is the left-hand side of (25). Therefore $E(\rho_0)$ given in (23) increases monotonically with ρ_0 and the theorem is proved.

Remark. A theorem by Hall and Post (1967) gives a lower bound for the N-particle problem in terms of the ground state of a modified two-particle problem. Using their theorem the following lower bound is obtained

$$\varepsilon_{\mathrm{HP}} = -(m/16\hbar^2) N^2 (N-1) I^2. \tag{26}$$

This lower bound is worse than the one given by theorem 8.

Remark. It would be nice to extend theorem 2 to the N-particle case and thus obtain a *necessary condition for the existence of an antisymmetric bound state.* But we were unable to obtain the corresponding analytic solution of (21). Numerical solutions can be obtained.

Remark. It is easy to show that the solution of the N-body problem given by (21) tends to a finite limit when $a \to \infty$. We think that this limit is $\frac{1}{4} N^2 E^{(2)}(a \to \infty)$ for N even and $\frac{1}{4}(N^2 - 1) E^{(2)}(a \to \infty)$ for N odd, where $E^{(2)}$ is the energy of a particle moving in the potential $B[\delta(x-a) + \delta(x+a)]$. We conjecture that the values given above are *exact.*

We shall now extend theorem 4 to the N-particle case.

Theorem 10. If the interparticle potential $V(x_i - x_j)$ is not globally attractive then the many-body Hamiltonian

$$H = H_0 + \sum_{i>j} \lambda V(|x_i - x_j|) \tag{27}$$

has no bound states for sufficiently small λ.

Proof. The proof follows the proofs of theorems 4 and 8. We prove that the ground-state energy of H is greater than the ground-state energy of

$$H_\delta = H_0 - \lambda I \sum_{i>j} \delta(|x_i - x_j| - x_-(\lambda)) + \lambda I' \sum_{i>j} \delta(|x_i - x_j| - x_+(\lambda)). \tag{28}$$

Removing the centre-of-mass energy from (28) we obtain

$$H_\delta - E_{\mathrm{CM}} = \sum_{i<j=1}^{N} \left(-\frac{\hbar^2}{2mN} (\nabla_{r_i} - \nabla_{r_j})^2 - \lambda I \sum_{i>j=1} \delta(|x_i - x_j| - x_-(\lambda)) \right.$$
$$\left. + \lambda I' \sum_{i>j} \delta(|x_i - x_j| - x_+(\lambda)) \right). \tag{29}$$

Now the Hall and Post (1967) theorem gives as a lower bound for (29) the energy of the two-body Hamiltonian:

$$H_{\mathrm{HP}} = -(N-1)\left(\frac{\hbar}{2m}\nabla_\rho^2 - \frac{N}{2}[-\lambda I\delta(\sqrt{2}\rho - x_-(\lambda)) + \lambda I'\delta(\sqrt{2}\rho - x_+(\lambda))]\right) \tag{30}$$

where $\rho = (x_1 - x_2)/\sqrt{2}$; hence for λ sufficiently small (27) has no bound state.

Remark. It is not too difficult to prove the existence of a bound state for the N-particle problem if $\int_{-\infty}^{\infty} V(x)\,\mathrm{d}x = 0$ using the technique described by Coutinho *et al* (1983).

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